Parallelization of Fully Distributed dense Matrix-Matrix Multiplication (1)

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Lessons for Parallelization of Matrix-Matrix Multiplications

- Lesson 1
  - This lesson.
  - Easy to parallelize. It needs 30 minutes or so.
  - No communication is needed.
- Lesson 2
  - Next lesson.
  - Medium level. It needs one hour or so.
  - 1-to-1 communications are used.

What is matrix-matrix multiplication?

The basic operation that can improve performance by code optimization.

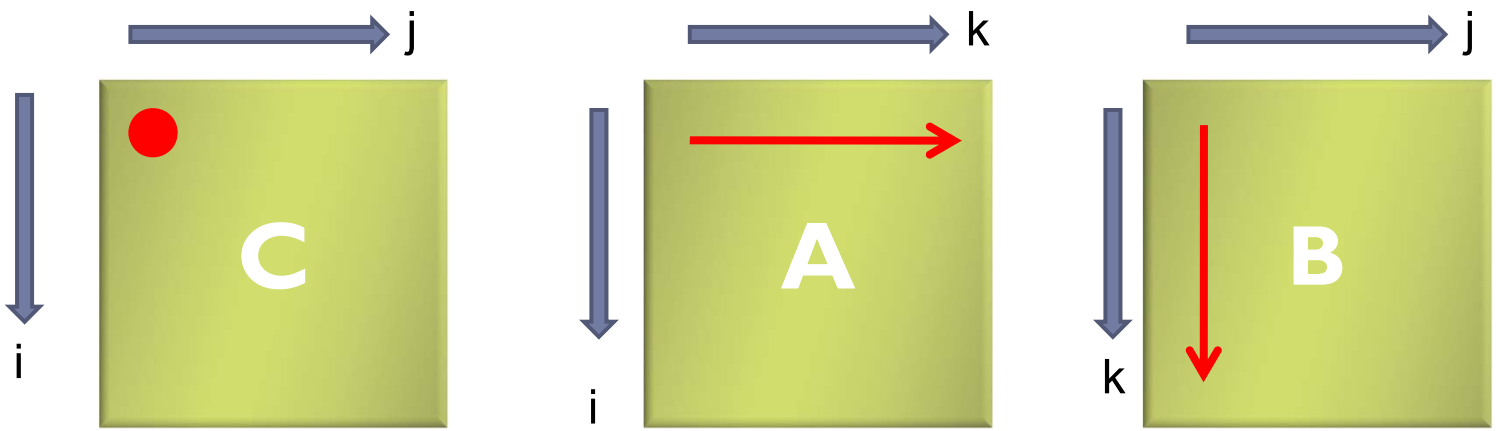
Dense Matrix-Matrix Multiplication

- A dense matrix-matrix multiplication C = A B is utilizing a benchmark for compilers and computer systems.
  - Reason 1: Big impact of performance depends on implementations.
  - Reason 2: Easy to understand. It can also implement codes easily.
  - Reason 3: It represents characteristics of scientific and technology computations.
    1. There is a large <continuous> loop.
    2. It accesses <big data> without cache memory in simple implementation.
    3. If 2, it is memory intensive computation, which accesses memory frequently.

A Simple Implementation (C Language)

- An implementation:

```
for (i=0; i<n;  i++)
  for (j=0; j<n; j++)
    for (k=0; k<n; k++)
      C[i][j] += A[i][k] *B[k][j];
```



Optimization Methods for Matrix-matrix Multiplication (MMM)

- A Matrix-matrix multiplication:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \ (i, j = 1, 2, \ldots, n)$$

can be optimized by the followings:

1. Loop Exchange Method:
   - Exchange 3-nested loops of MMM to perform continuous access.
2. Blocking (Tiling) Method:
   - Implement codes to reuse of data in a partial part of matrices in cache memory.

Loop Exchange Method (C Language)

- The loop of MMM forms the following 3-nested loop:

```
for(i=0;  i<n;  i++)  {
   for(j=0;  j<n;  j++)  {
      for(k=0; k<n; k++)  {
           c[ i ][ j ] = c[ i ][ j ] + a[ i ][ k ] * b[ k][ j ];
      }
   }
}
```

  - Although we exchange the outer loops, result of computation is not changed with respect to inner computation.

    → Hence we have 6 ways to exchange the loop.

Loop Exchange Method (Fortran Language)

- The loop of MMM forms the following 3-nested loop:

```
do i=1,  n
   do j=1,  n
      do k=1,  n
           c( i ,  j ) = c( i,  j) + a( i , k ) * b( k ,  j )
      enddo
   enddo
enddo
```

- If we exchange the outer loops, results of computation do not change with respect to inner computation.
  → Hence we have 6 ways to exchange the loop.

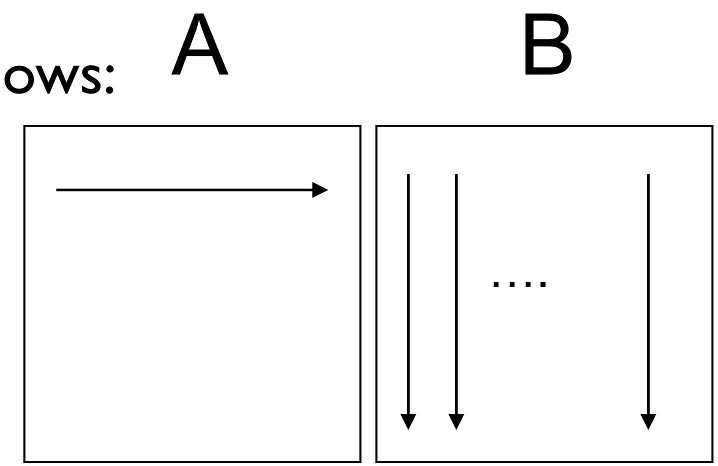
Classification of MMM

- There are three classifications for MMM according to memory access pattern.
  1. Inner-product form
     It is same as <dot products of vectors> for access pattern of the inner computation.
  2. Outer-product form
     It is same as <outer products of vectors> for access pattern of the inner products.
  3. Middle-product form
     It is hybrid form between inner-product and outer-product forms.

The inner-product form of MMM (C Language)

- Inner-product form
  - Implementation with ijk, jik loops as follows:

```
for (i=0; i<n; i++) {
   for (j=0; j<n; j++) {
      dc = 0.0;
      for (k=0; k<n; k++){
         dc = dc + A[ i ][ k ] * B[ k ][ j ];
      }
      C[ i ][ j ]= dc;
   }
}
```



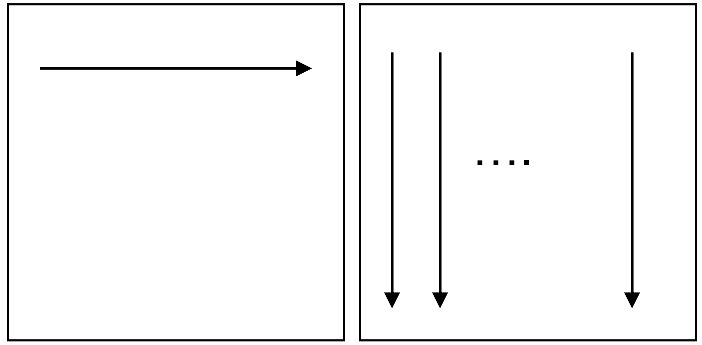
* With accesses for row-size and column-wise:
→ Performance goes down between languages that provide row-wise and column-wise allocations.
One of solutions：
Transpose array for A or B.

*Here after, we denote implementation with order of loop induction variables from the outer loop. For example, the above code is <ijk loop>.

The inner-product form of MMM (Fortran Language)

- Inner-product form
  - Implementation with ijk, jik loops as follows:

A B



```
do i=1, n
  do j=1, n
    dc = 0.0d0
    do k=1, n
      dc = dc + A( i , k ) * B( k , j )
    enddo
    C( i , j ) = dc
  enddo
enddo
```

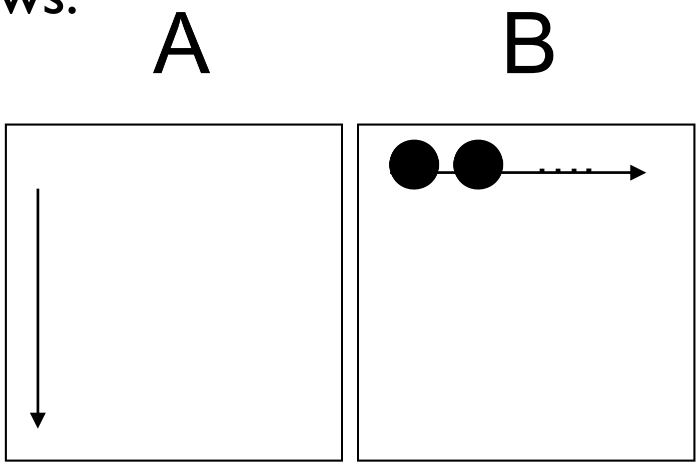
*Here after, we denote implementation with order of loop induction variables from the outer loop. For example, the above code is <ijk loop>.

* With accesses for row-size and column-wise:
→ Performance goes down between languages that provide row-wise and column-wise allocations.
One of solutions：
Transpose array for A or B.

The outer-product form of MMM (C Language)

- Outer-product form
  - Implementation with kij, kji loops as follows:

```
for (i=0;  i<n;  i++) {
   for (j=0;  j<n;  j++) {
       C[ i ][ j ] = 0.0;
   }
}
for (k=0;  k<n;  k++) {
  for (j=0;  j<n;  j++) {
     db = B[ k ][ j ];
     for (i=0;  i<n;  i++) {
      C[ i ][ j ]= C[ i ][ j ]+ A[ i ][ k ]* db;
     }
  }
}
```



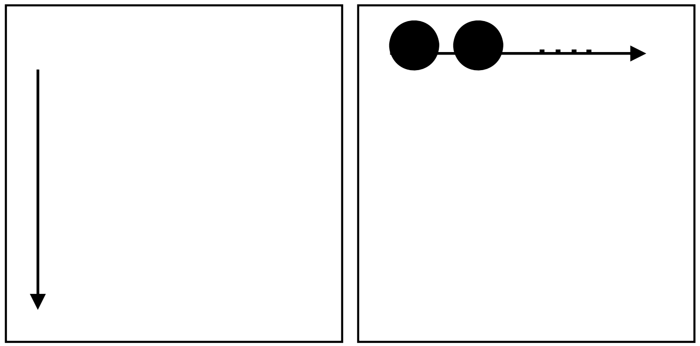
* In kji loop, main access direction is column-wise.
→ It is good for language that provides column-wise array allocation.
(Fortran)

The outer-product form of MMM (Fortran Language)

- Outer-product form
  - Implementation with kij, kji loops as follows:

```
do i=1, n
   do j=1, n
      C( i , j ) = 0.0d0
   enddo
enddo
do k=1, n
  do j=1, n
     db = B( k ,  j )
     do i=1, n
       C( i , j ) = C( i , j )+ A( i , k ) * db
     enddo
  enddo
enddo
```

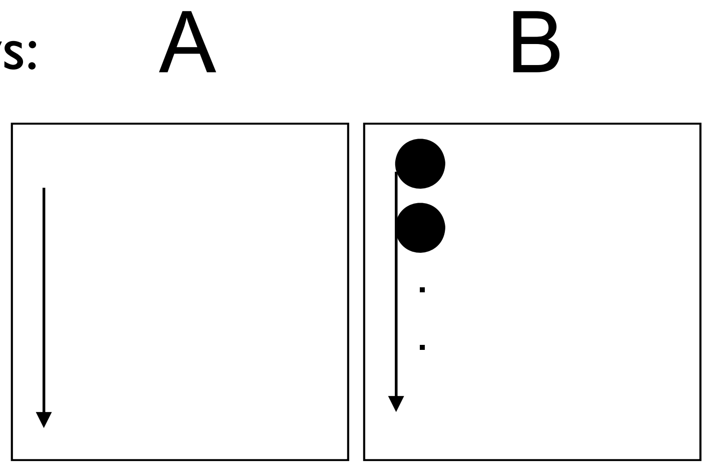
A B



* In kji loop, main access direction is column-wise.
→ It is good for language that provides colmn-wise array allocation.
(Fortran)

The middle-product form of MMM (C Language)

- Middle-product form
  - Implementation with ikj, jki loops as follows:



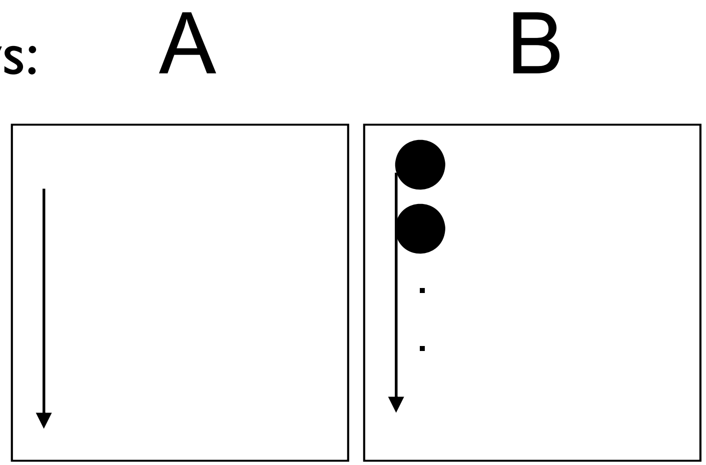

```
for (j=0; j<n; j++) {
   for (i=0; i<n; i++) {
     C[ i ][ j ] = 0.0;
   }
   for (k=0; k<n; k++) {
     db = B[ k ][ j ];
     for (i=0; i<n; i++) {
      C[ i ][ j ] = C[ i ][ j ] + A[ i ][ k ] * db;
     }
   }
}
```

* In jki loop, all access directions are column-wise.
→ It is the best for language that provides column-wise array allocation. (Fortran)

The middle-product form of MMM (Fortran Language)

- Middle-product form
  - Implementation with ikj, jki loops as follows:

```
do j=1, n
   do i=1, n
      C( i , j ) = 0.0d0
   enddo
   do k=1, n
     db = B( k , j )
     do i=1, n
        C( i , j ) = C( i , j ) + A( i , k ) * db
     enddo
   enddo
enddo
```



* In jki loop, all access directions are column-wise.
→ It is the best for language that provides column-wise array allocation. (Fortran)

Execution of sample program (Dense matrix-matrix multiplication)

Note: sample program of dense matrix-matrix multiplication

- Common file name of C/Fortran languages:

  Mat-Mat-fx.tar

- Modify queue name from lecture to lecture7 in job script file mat-mat.bash. Then type "pjsub".
  - lecture : Queue in out of time of this lecture.
  - lecture7 Queue in time of this lecture.

Execute sample program of dense matrix-matrix multiplication

- Type followings in command line:

```
$ cp /home/z30082/Mat-Mat-fx.tar ./
$ tar xvf Mat-Mat-fx.tar
$ cd Mat-Mat
```

- Choose the follows:

`$ cd C` : For C language.

`$ cd F` : For Fortran language.

- The follows are common:

```
$ make
$ pjsub mat-mat.bash
```

- After finishing the job, type the follow:

```
$ cat mat-mat.bash.oXXXXXX
```

Output of sample program of dense matrix-matrix multiplication (C Language)

- If the run is successfully ended, you can see the follows:

N = 1000

Mat-Mat time = 0.209609 [sec.]

9541.570931 [MFLOPS]

OK!

It is established 9.5GFLOPS with one core.

Output of sample program of dense matrix-matrix multiplication (Fortran Language)

- If the run is successfully ended, you can see the follows:

```
NN  =  1000
Mat-Mat time[sec.] =  0.204734672959827
MFLOPS =  9768.741003580422
 OK!
```

It is established 9.7GFLOPS with one core.

Explanation of sample program (C Language)

- You can change size of matrix by the number:

  #define N 1000

- By setting 1 in the follow "0", result of matrix-matrix multiplication is verified:

  #define DEBUG 0

- Specification of MyMatMat function
  - Return result of A times B with size of [N][N] of **double** by setting C with size of [N][N] of **double**.

Explanation of sample program (Fortran Language)

- You can find declaration of size of dimension N in the following file:
  mat-mat.inc

- Variable of the size of dimension is NN, such as:
  integer NN
  parameter (NN=1000)

Homework 4

- Parallelize MyMatMat function.
  You can use the following parameter for debugging.
  - #define N 192
  - #define DEBUG 1

- Whole elements of matrices A, B, and C, that are size of N × N, can be allocated in each PE redundantly. (c.f. Strategy of parallelization)

Note: Parallelization

- In this sample program, we use a test matrix with that all elements are set to "1" for A and B. Then we compare theoretical result, that is: all elements of C are N. Please use function of verification for your debug.

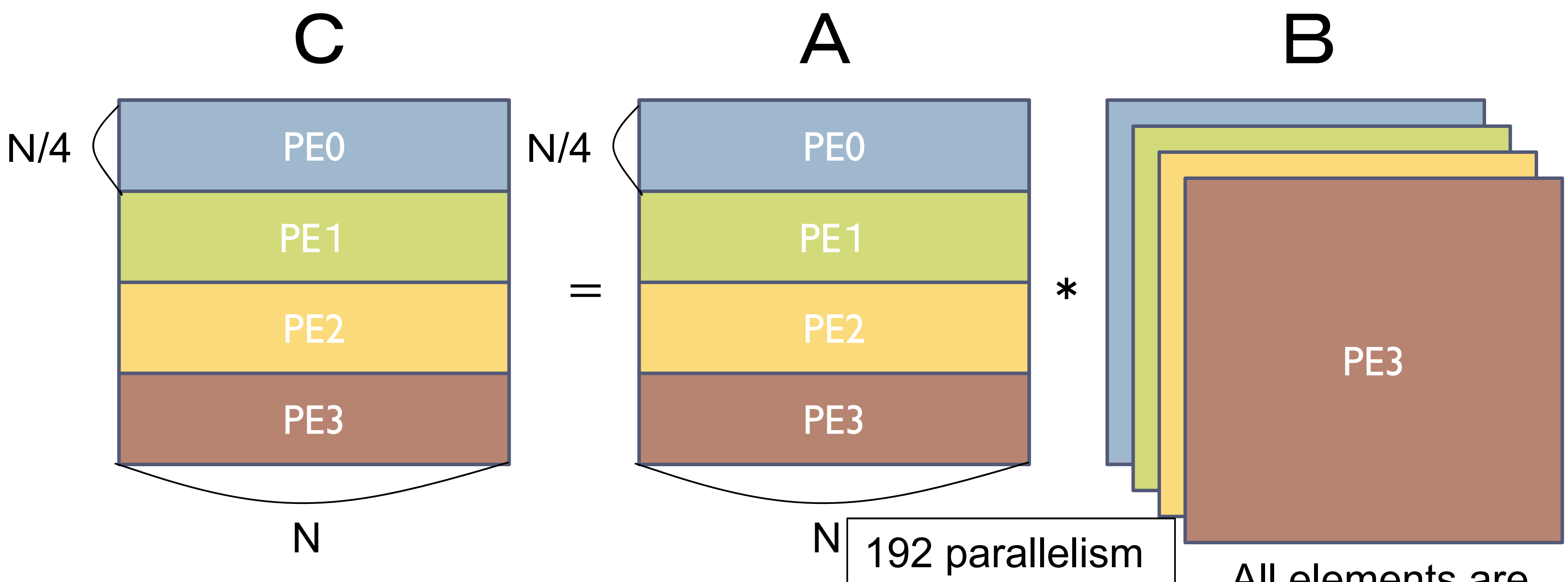
  Note:

  You need also parallelization for the verification routine.

  (c.f. Sample program of matrix-vector multiplication.)

Hints of parallelization

- Use the following data distribution to do easy implementation:



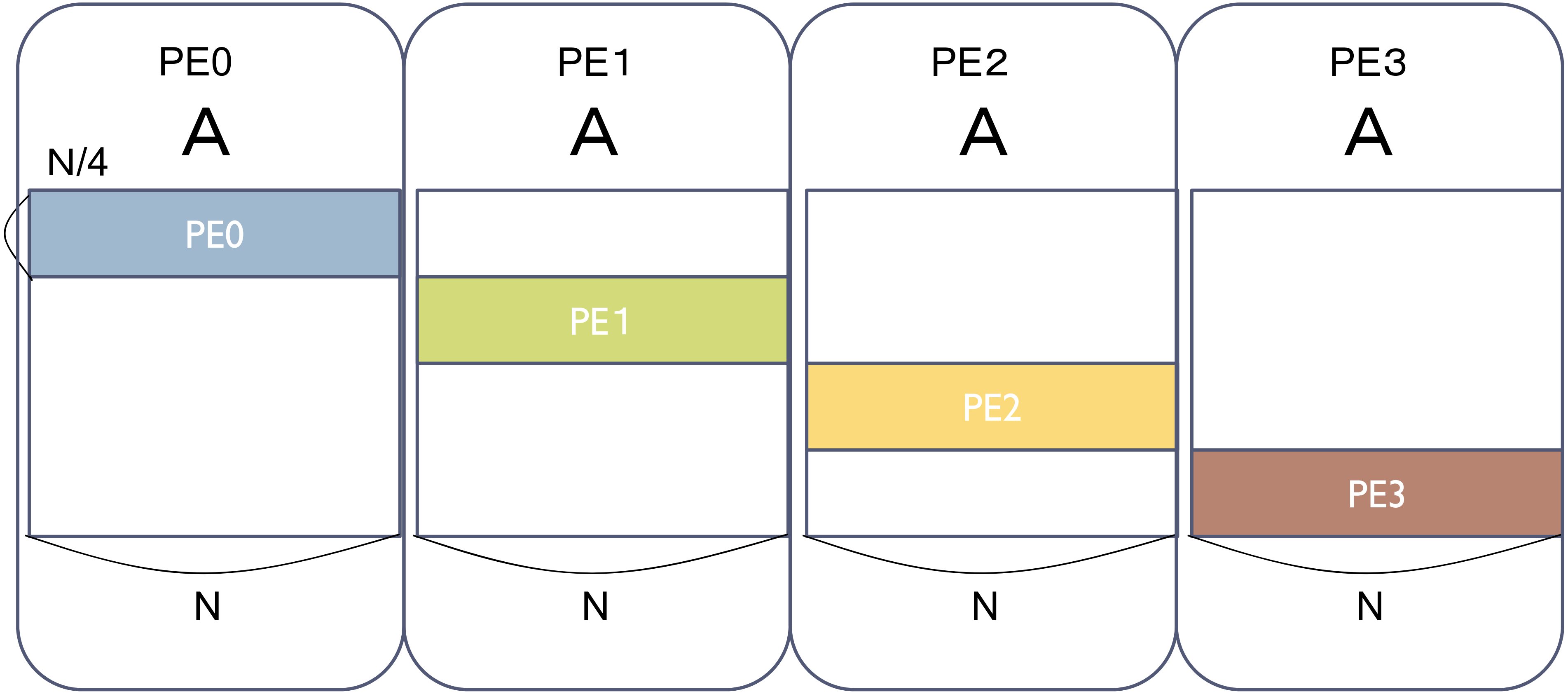
192 parallelism can be used in this lecture.

All elements are allocated redundantly in each PE.

- No communication is needed.
- It can be parallelized as same as matrix-vector multiplication.

Confirmation: Allocation of array in viewpoint of each PE

- Use "partial" part of arrays in each PE
  although it allocates whole of size of arrays [N][N].



Note: performance issue by implementation

- If you use global variables for loop induction variables, you may obtain poor performance.
- Use it by local variables, or literal values, such as 100.

Use local variables

```
for (i=i_start; i<i_end; i++) {
      …
      …
}
```

Homework and Lessons

1. [Homework4] Parallelize sample code of dense matrix-matrix multiplication. You can use redundant allocation of arrays for matrix A, B, and C for initial data distribution.
2. Make a hybrid MPI/OpenMP code, then evaluate its performance by using several executions with respect to MPI processes and OpenMP threads in environments of lecture. Find condition that pure MPI is the fastest by using results of the evaluation.